if x is less than $7° 15'$, then to five decimals, $\log_{10} x = \log_{10} \sin x + \frac{1}{3} \log_{10} \sec x = \log_{10} \tan x - \frac{2}{3} \log_{10} \sec x$.

Solution by the Proposer.

$$y = \log \frac{x}{\sin x}, \quad \text{or} \quad e^y = \frac{x}{\sin x}; \quad \frac{dy}{dx} = \frac{1}{x} - \cot x.$$

$$z = \log \sec x, \quad \text{or} \quad e^z = \sec x; \quad \frac{dz}{dx} = \tan x.$$

Hence,

$$\frac{dy}{dx} = \tan x \frac{dy}{dz}.$$

Also

$$z - y = \log \frac{\tan x}{x}, \quad \text{or} \quad e^{z-y} = \frac{\tan x}{x}.$$

$$\tan x \frac{dy}{dz} = \frac{1}{x} - \cot x; \quad \tan^2 x \frac{dy}{dz} = \frac{\tan x}{x} - 1, \quad \frac{dy}{dz} = \frac{e^{z-y} - 1}{e^{2z} - 1} = \frac{e^{-y} - e^{-z}}{e^z - e^{-z}}.$$

$$z = 0 = y \quad \text{when} \quad x = 0.$$

$$\left.\frac{dy}{dz}\right]_{z \doteq 0} = \frac{0}{0} = \left.\frac{-\left.\frac{dy}{dz}\right] e^{-y} + e^{-z}}{e^z + e^{-z}}\right]_{z \doteq 0}; \quad \left.\frac{dy}{dz}\right]_{z \doteq 0} = \frac{-\left.\frac{dy}{dz}\right]_{z \doteq 0} + 1}{2}; \quad \left.\frac{dy}{dz}\right]_{z \doteq 0} = \frac{1}{3}.$$

$$\left.\frac{d^2y}{dz^2}\right]_{z \doteq 0} = \frac{1}{(e^z + e^{-z})^2}\left[2 - e^{-y}\left([e^z - e^{-z}]\frac{dy}{dz} + [e^z + e^{-z}]\right)\right]; \quad \left.\frac{d^2y}{dz^2}\right]_{z \doteq 0} = \frac{0}{0}.$$

$$\left.\frac{d^2y}{dz^2}\right]_{z \doteq 0} = \frac{1}{2(e^{2z} - e^{-2z})}\left[- e^{-y}\left[(e^z - e^{-z})\frac{d^2y}{dz^2} - (e^z - e^{-z})\left(\frac{dy}{dz}\right)^2 + (e^z - e^{-z})\right]\right]_{z \doteq 0}$$

$$= \frac{-e^{-y}}{2(e^z + e^{-z})}\left[\frac{d^2y}{dz^2} - \left(\frac{dy}{dz}\right)^2 + 1\right]_{z \doteq 0}; \quad \left.\frac{d^2y}{dz^2}\right]_{z \doteq 0} = -\frac{8}{45}.$$

$$y = \frac{z}{3} - \frac{8}{45}z^2 \cdots, \quad z - y = \frac{2}{3}z + \frac{8}{45}z^2 \cdots,$$

or

$$\log x - \log \sin x = \frac{1}{3}\log \sec x - \frac{8}{45}(\log \sec x)^2,$$

$$\log_{10} x - \log_{10} \sin x = \frac{1}{3}\log \sec x - \frac{8}{45M}(\log \sec x)^2.$$

Similarly,

$$\log_{10} \tan x - \log_{10} x = \frac{2}{3}\log \sec x + \frac{8}{45M}(\log \sec x)^2,$$

$$\frac{8}{45M}(\log_{10} \sec x)^2 = .000005 \text{ if } \log_{10} \sec x = .0034948,\ x = 7° 15',$$

$$\frac{8}{45M} = .4 \text{ nearly}, \quad = (.40926), \quad = \frac{70}{171} \text{ very closely}.$$

This theorem enables one to dispense without inconvenience with the usual table for small angles. It is well to take $\log_{10}$ c.m. $1' = 6.46372\frac{2}{3} - 10$ in order to account for odd thirds. The theorem gives correct results in most instances for values of θ up to about $10°$. For $\theta = 10°$ there is an error of one unit in the fifth place of decimals.

369. Proposed by I. A. BARNETT, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

I. Solution by A. M. Harding, University of Arkansas.

Let

$$S_n = dx \cdot \log a + dx \cdot \log (a + dx) + dx \cdot \log (a + 2dx) + \cdots + dx \cdot \log [a + (n-1)dx].$$

Then

$$I = \int_a^b \log x dx = \lim_{n=\infty} S_n, \text{ where } ndx = b - a.$$

Now

$$\log (a + dx) = \log a + \frac{1}{a} dx - \frac{1}{2a^2} dx^2 + \frac{1}{3a^3} dx^3 - \frac{1}{4a^4} dx^4 + \cdots$$

$$\log (a + 2dx) = \log a + \frac{1}{a} 2dx - \frac{1}{2a^2} 2^2dx^2 + \frac{1}{3a^3} 2^3dx^3 - \frac{1}{4a^4} 2^4dx^4 + \cdots$$

$$\log [a + (n-1)dx] = \log a + \frac{1}{a}(n-1)dx - \frac{1}{2a^2}(n-1)^2dx^2 + \frac{1}{3a^3}(n-1)^3dx^3 - \frac{1}{4a^4}(n-1)^4dx^4 + \cdots.$$

Hence,

$$S_n = \log a \cdot ndx + \frac{1}{a}[1 + 2 + 3 + 4 + \cdots + (n-1)]dx^2 - \frac{1}{2a^2}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]dx^3$$
$$+ \frac{1}{3a^3}[1^3 + 2^3 + 3^3 + \cdots + (n-1)^3]dx^4 - \frac{1}{4a^4}[1^4 + 2^4 + 3^4 + \cdots + (n-1)^4]dx^5 + \cdots$$
$$= (b-a)\log a + \frac{1}{a}(n^2/2 - n/2)dx^2 - \frac{1}{2a^2}(n^3/3 - n^2/2 + n/6)dx^3$$
$$+ \frac{1}{3a^3}(n^4/4 - n^3/2 + n^2/4)dx^4 - \frac{1}{4a^4}(n^5/5 - n^4/2 + n^3/3 - n/30)dx^5$$
$$+ \frac{1}{5a^5}(n^6/6 - n^5/2 + 5n^4/12 - n^2/12)dx^6 - \cdots;$$

$$\therefore I = \lim S_n = (b-a)\log a + \frac{(b-a)^2}{2a} - \frac{(b-a)^3}{2 \cdot 3a^2} + \frac{(b-a)^4}{3 \cdot 4a^3} - \frac{(b-a)^5}{4 \cdot 5a^4} + \frac{(b-a)^6}{5 \cdot 6a^5} - \cdots$$
$$= (b-a)\log a + \left(1 - \frac{1}{2}\right)\frac{(b-a)^2}{a} - \left(\frac{1}{2} - \frac{1}{3}\right)\frac{(b-a)^3}{a^2} + \left(\frac{1}{3} - \frac{1}{4}\right)\frac{(b-a)^4}{a^3}$$
$$+ \left(\frac{1}{4} - \frac{1}{5}\right)\frac{(b-a)^5}{a^4} - \cdots$$
$$= (b-a)\log a + (b-a)[-(1-b/a) - 1/2(1-b/a)^2 - 1/3(1-b/a)^3 - 1/4(1-b/a)^4 - \cdots]$$
$$+ a\left[-\frac{1}{2}\left(\frac{b-a}{a}\right)^2 + \frac{1}{3}\left(\frac{b-a}{a}\right)^3 - \frac{1}{4}\left(\frac{b-a}{a}\right)^4 + \frac{1}{5}\left(\frac{b-a}{a}\right)^5 - \cdots\right]$$
$$= (b-a)\log a + (b-a)\log\left[1 - \left(1 - \frac{b}{a}\right)\right] + a\left[\log\left\{1 - \left(1 - \frac{b}{a}\right)\right\} + \frac{b-a}{a}\right]$$
$$= (b-a)\log a + (b-a)\log\frac{b}{a} + a\log\frac{b}{a} + b - a$$
$$= (b-a)\log a + b\log\frac{b}{a} + b - a$$
$$= b\log b - a\log a + b - a.$$

II. Solution by the Proposer.

Using the method of division indicated in the diagram, we have,

$$q^n = b \qquad \text{and} \qquad n = \log\frac{b}{a} - \log q.$$

Hence,

$$\int_a^b \log x dx = \lim_{q \doteq 1} [a(q-1)\log a + aq(q-1)\log aq + \cdots + aq^{n-1}(q-1)\log aq^{n-1}],$$

$$= \lim_{q \doteq 1} a(q-1)[(1+q+\cdots+q^{n-1})\log a + (q+2q^2+\cdots+(n-1)q^{n-1})\log q],$$

$$= \lim_{q \doteq 1} a(q-1)\left[\left(\frac{q^n-1}{q-1}\right)\log a + \left\{\frac{q-nq^n+(n-1)q^{n+1}}{(1-q)^2}\right\}\log q\right],^*$$

$$= \lim_{q \doteq 1} a\left[(q^n-1)\log a + \frac{q(1-q^n)}{q-1}\log q + nq^n \log q\right],$$

$$= \lim_{q \doteq 1} a\left[\left(\frac{b-a}{a}\right)\log a + \frac{q}{q-1}\left(\frac{a-b}{a}\right)\log q + \frac{b}{a}\log\frac{b}{a}\right],$$

$$= b\log b - a\log a + b - a.$$

Also solved similarly by P. PENALVER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1+\csc\phi)}$.

SOLUTION BY PAUL CAPRON, Annapolis, Maryland.

If the projectile is fired at an elevation θ, its trajectory may be represented by $x = vt\cos\theta$, $y = vt\sin\theta - \frac{1}{2}gt^2$, or by

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta.$$

To find the envelope of all such trajectories, we have

$$0 = x\sec^2\theta - \frac{gx^2}{v^2}\sec^2\theta\tan\theta, \qquad \text{or} \qquad \tan\theta = \frac{v^2}{gx};$$

whence

$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2}$$

is the envelope.

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* To sum

$$q + 2q^2 + 3q^3 + \cdots + (n-1)q^{n-1} = q(1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}) = q\Sigma$$

$$\Sigma = 1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}$$

$$-q\Sigma = -q - 2q^2 - \cdots - (n-2)q^{n-2} - (n-1)q^{n-1}$$

$$\therefore (1-q)\Sigma = 1 + q + q^2 + \cdots + q^{n-2} - (n-1)q^{n-1}$$

$$= \frac{q^{n-1}-1}{q-1} - (n-1)q^{n-1}$$

$$\therefore q\Sigma = \frac{q - nq^n + (n-1)q^{n+1}}{(1-q)^2}.$$